3.16.1. Derived Rule Problems

For each of the following deductive systems, show that one or more of our deductive rules are **derived rules** in that system (by constructing a deduction of that rule, in that system).

1. Show that the rule **MP** can be treated as a derived rule in our Chapter Three deductive system, by providing a deduction of the argument "$(P \rightarrow Q)$. P $\therefore$ Q" that doesn't use MP.

2. Show that the rule **MT** can be treated as a derived rule in our Chapter Three deductive system, by providing a deduction of the argument "$(P \rightarrow Q)$. ~Q $\therefore$ ~P" that doesn't use MT.

3. The deductive system **DS1** has only ID, plus the the rules ∧–, ∧+, ∨–, ∨+, and "Negated Conditional" ("~→").

Negated Conditional (~→)

$$\frac{\sim(\bullet \rightarrow \blacktriangle)}{\therefore\ (\bullet \wedge \sim\blacktriangle)} \qquad \frac{(\bullet \wedge \sim\blacktriangle)}{\therefore\ \sim(\bullet \rightarrow \blacktriangle)}$$

Thanks to **~→**, any **CD** from our system of deduction can be converted into an ID in DS1.[1]

> 3a. Show that **MT** is a derived rule in DS1.
>
> 3b. Show that **MP** is a derived rule in DS1.

[1] Because DS1 has **ID** and the rules ∨ – and ∨+, the rules R, ~–, and ~+ are already derived rules in this system – as shown in 2.40.1 Problems 2, 3, and 5.

4. The deductive system **DS2** is like our Chapter Three deductive system except that it lacks the rule $\vee-$ and instead has the rule **Separation of Cases (SC)**.

Separation of Cases (SC)

$$\begin{array}{c} (\bullet \vee \heartsuit) \\ (\bullet \rightarrow \blacktriangle) \\ (\heartsuit \rightarrow \blacktriangle) \\ \hline \blacktriangle \end{array}$$

So the following argument is an example of (SC).

$$\begin{array}{c} 1.\ (P \vee Q) \\ 2.\ (P \rightarrow R) \\ 3.\ (Q \rightarrow R) \\ \hline \therefore\ R \end{array}$$

4a. Show that the above argument is deducible in the Chapter Three deductive system (and hence that **SC** can act as a derived rule in the Chapter Three system).

4b. Show that the following argument is deducible in the system **DS2** (and hence that $\vee-$ is a derived rule in **DS2**) using only **CD**, **ID**, **R**, and **SC**.

$$\begin{array}{c} 1.\ (P \vee Q) \\ 2.\ {\sim}P \\ \hline \therefore\ Q \end{array}$$

5. The following argument is an instance of the rule **Double Disjunction** (DD), discussed in Chapter Two.

$$\begin{array}{l} 1.\ (P \vee Q) \\ 2.\ (P \vee \sim Q) \\ \hline \therefore\ P \end{array}$$

Show that Double Disjunction is a derived rule, by providing a deduction of the above argument using only **~+**, ∨**–**, **SC**, and **CD**.

6.

Separation of Cases (SC)

$$\begin{array}{c} (\bullet \vee \heartsuit) \\ (\bullet \rightarrow \blacktriangle) \\ (\heartsuit \rightarrow \blacktriangle) \\ \hline \blacktriangle \end{array}$$

Negated Conjunction (~∧)

$$\begin{array}{c} \sim \bullet \\ \hline \therefore\ \sim(\bullet \wedge \blacktriangle) \end{array}$$

Provide a deduction for the following argument using only **SC**, ~∧, and **CD**.

$$\begin{array}{c} (\sim P \vee \sim Q) \\ \hline \therefore\ \sim(P \wedge Q) \end{array}$$

7. Note that the argument "$(P \rightarrow (P \rightarrow Q))\ \therefore\ (P \rightarrow Q)$" is valid. We can use the rule Separation of Cases to **explain** the validity of this rule **deductively**: from the sentence "$(P \rightarrow (P \rightarrow Q))$," when accompanied by the sentences "$(P \vee \sim P)$" and "$(\sim P \rightarrow (P \rightarrow Q))$," the conclusion "$(P \rightarrow Q)$" follows as an instance of Separation of Cases.

1. $(P \vee \sim P)$
2. $(P \rightarrow (P \rightarrow Q))$
3. $(\sim P \rightarrow (P \rightarrow Q))$

$\therefore$ **$(P \rightarrow Q)$** 1, 2, 3, SC

But the other two premises, "$(P \vee \sim P)$" and "$(\sim P \rightarrow (P \rightarrow Q))$," are **theorems** provable without appeal to premises[2] – in effect, built in the deductive system. In that sense, from the deductive system alone "$(P \rightarrow (P \rightarrow Q))$" entails "$(P \rightarrow Q)$" by Separation of Cases.

For each of the following arguments, use **Separation of Cases** plus theorems to deductively explain the validity of that argument.

7a. $(P \vee Q)$. $(P \rightarrow Q)\ \therefore\ Q$

7b. $(\sim P \rightarrow P)\ \therefore\ P$

7c. $(P \rightarrow \sim(P \wedge Q))\ \therefore\ \sim(P \wedge Q)$

7d. $(P \rightarrow (Q \rightarrow \sim P))\ \therefore\ (Q \rightarrow \sim P)$

7e. $((P \rightarrow Q) \rightarrow P)\ \therefore\ P$

[2] "$(P \vee \sim P)$" is theorem **T2.2** from 2.41.1, and "$(\sim P \rightarrow (P \rightarrow Q))$" is theorem **T3.3b** from 3.13.1 C.

8. Using the rule **Double Conditional** (**DC**), in two forms, we can remove Indirect Deduction and a number of rules.

Double Conditional (DC)

(I)	(II)
$(\bullet \rightarrow \blacktriangle)$	$(\sim\bullet \rightarrow \blacktriangle)$
$(\bullet \rightarrow \sim\blacktriangle)$	$(\sim\bullet \rightarrow \sim\blacktriangle)$
$\therefore\ \sim\bullet$	$\therefore\ \bullet$

MT, for example, can be replaced with a deduction of the following form.

1. (P → Q)
2. ~Q

Get: ~P

Get: (P → ~Q)

3.	P	ACD
4.	~Q	2, R
5.	(P → ~Q)	3, 4, CD
6.	~P	1, 5, DC (I)

Provide **deductions** of each of the following arguments using only **DC**, **R**, **∧+**, **∧−**, **∨+**, **∨−**, and **CD**.

1. (P → Q) . P ∴ Q

(*Hint: use DC (II) and the MT deduction given above.*)

2. P ∴ ~~P

3. ~~P ∴ P

4. ~P ∴ ~(P ∧ Q)